JOURNAL OF COMBINATORIAL THEORY, Series B 55, 118–124 (1992)

Submaps of Maps. II. Cyclically k-Connected Planar Cubic Maps*

EDWARD A. BENDER†

AND

ZHI-CHENG GAO

Department of Mathematics, University of California, San Diego, La Jolla, California 92093

WILLIAM D. MCCUAIG

Department of Mathematics, University of Victoria, Victoria, British Columbia, V8W 2Y2, Canada

AND

L. BRUCE RICHMOND‡

Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada

Communicated by the Editors

Received February 21, 1989

We show that, for $1 \leqslant k \leqslant 5$, if a map M can be contained in a k-cycle so as to preserve k-connectivity, then almost all cyclically k-connected cubic graphs with n vertices contain at least cn copies of M as submaps for some constant $c(M)$. By using Walther's construction for maps without Hamiltonian cycles, we obtain an M with which we prove that almost no cyclically k-connected cubic maps with n vertices have a path of length greater than cn, where c is a constant. © 1992 Academic Press, Inc.

* The U. S. Government's right to retain a nonexclusive royalty-free license in and to the copyright covering this paper, for governmental purposes, is acknowledged.

† Research supported by the NSA under Grant MDA904-88-H-2015.

‡ Research supported by the NSERC under Grant A-4067.

0095-8956/92 $5.00

1. Introduction

The study of longest paths in planar cubic graphs has a relatively long history in graph theory. We say a graph is cyclically k-connected if it is not possible to break the graph into two separate parts, each containing a cycle, by the removal of fewer than k edges. From now on we limit our attention to (planar) maps: graphs that have been embedded in the plane. In 1884 P. G. Tait [6] published the conjecture that every cyclically 3-connected cubic map contains a Hamilton cycle; i.e., a cycle containing all the vertices. In 1946 W. T. Tutte [7] constructed a counterexample and in 1960 [8] he constructed a cyclically 4-connected cubic map without a Hamilton cycle. In 1965 H. Walther [10, 11] constructed a 5–connected counterexample. Descriptions of these graphs and further discussion can be found in B. Grünbaum [3, Chap. 17]. Since any regular graph without 3–cycles or 4–cycles contains at least twelve 5-gons, this brought to an end attempts to modify Tait's conjecture by increasing the cyclic connectivity.

Tait made his conjecture because it implies the Four Color Theorem. It is well known that the Four Color Theorem can be reduced to the case of cyclically 3-connected cubic maps. If a cubic map has a Hamilton cycle, it can be 4-colored by using two colors inside the cycle and two others outside. In 1988 L. B. Richmond and N. C. Wormald [5] showed that to prove the Four Color Theorem, it suffices to show that the radius of convergence of the generating function for 4-colorable cyclically 3-connected cubic maps counted according to edges equals the radius for all cyclically 3-connected cubic maps. Unfortunately, they also showed that this weakened form of Tait's conjecture is false.

It is also well known that the Four Color Theorem can be reduced to coloring cyclically 4- or 5-connected cubic maps. In this paper we show that Richmond and Wormald's weakened form of Tait's conjecture is also false in these cases.

In this paper we adapt the methods of E. A. Bender, Z.-C. Gao, and L. B. Richmond [1] to submaps of cyclically k-connected cubic maps for $k=4$ and $k=5$, obtaining results like those Richmond and Wormald obtained for $k=2$ and $k=3$. One consequence of this is that for some $c<1$ the fraction of such n vertex maps with a path whose length exceeds cn is exponentially small, thereby disproving the weakened Tait conjecture. Our results are not as precise as those of Richmond and Wormald, but our proofs are simpler and require much less information about the generating functions for the maps being studied.

Since the cyclically 2- and 3-connected cubic maps were studied in [5], we will look only at 4- and 5-connected cubic maps; however, our methods can be used for the other cases, too.

2. Preliminaries

All limits of the form $\lim_{n \to \infty} f(n)$ are understood to be taken through those integers for which $f(n) \neq 0$.

It is sometimes more convenient to work with the planar duals of cubic maps which are triangulations. A cubic map is cyclically k-connected if and only if its dual triangulation has no j-cycle with vertices in both its interior and exterior for any $j < k$. We will call a map which has no such separating cycle k-simple. Thus, a 2-simple triangulation is just a triangulation, a 3-simple triangulation is a triangulation without multiple edges (i.e., a 3-connected triangulation), and a 4-simple triangulation is what Tutte [9] calls a simple triangulation (i.e., it is 3-connected and every triangle is a face). A 1-simple triangulation need not be a triangulation in the usual sense since loops may be present. A 4-simple quadrangulation is what is called a simple quadrangulation.

If F is a set of positive integers, an F-map is a map all of whose face valencies lie in F. We will be particularly concerned with k-simple $\{3, k\}$-maps. Two such maps are shown in Fig. 1 and 2 for $k = 5$ and $k = 4$, respectively. They have an important property: If the boundaries of faces P and Q are identified, one each, with the boundaries of two valency k faces, one each from any two k-simple maps, the result is still k-simple.

3. Smooth Growth

In this section we prove

Theorem 1. *If $k \leqslant 5$ and t_n is the number of cyclically k-connected cubic maps with n edges, then* $\lim_{n \to \infty} t_n^{1/n}$ *exists. Of course, t_n is also the number of k-simple triangulations.*

We need not be concerned with whether the maps are rooted or not since rooting introduces a factor of at most $4n$ into t_n. The cases $k = 1$ and $2 \leqslant k \leqslant 4$ follow from known asymptotic results [2, 9]. We prove $k = 5$ here, adapting the three step method of [1, Sect. 2].

Let s_n be the number of n edge 5-simple $\{3, 5\}$-maps where all but the root face is a triangle. To begin with, we will show that studying $\lim t_n^{1/n}$ is equivalent to studying $\lim s_n^{1/n}$.

Fix a 5-simple $\{3, 5\}$-map S_1 in which all faces except some non-root face f are triangles. Create a new map S_2 by identifying the root face boundary of Fig. 1 with the boundary of f of S_1. Any 5-simple map that is a triangulation except for having a root face of valency 5 can have its root face boundary identified with the boundary of P of S_2. Thus $s_n \leqslant t_{n+e}$ for some fixed e.

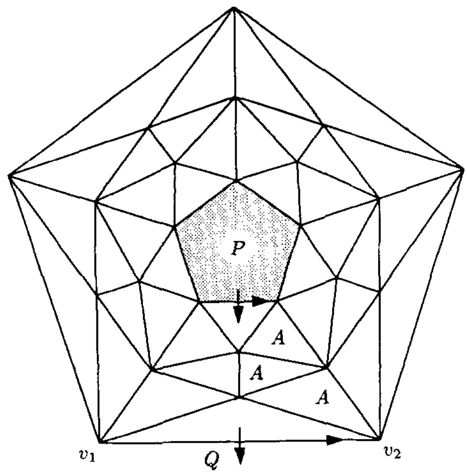

FIGURE 1

Conversely, let T be any 5-simple triangulation with more than four vertices. There must be some root face vertex with degree exceeding 3. Thus at least one-third of the 5-simple triangulations are such that the root face vertex not on the root edge has degree exceeding 3. Remove the two root face edges that are adjacent to this vertex to obtain a 5-simple $\{3, 5\}$-map. Thus $s_n \geqslant t_{n+2}$.

Let r be the radius of convergence of $\sum t_n x^n$. By the previous paragraphs, it is also the radius of convergence of $\sum s_n x^n$. Let $C_i > 1$ and $1 - r > \delta > 0$ be arbitrary. We will show:

Step 1. For some n, $s_n > C_1(r+\delta)^{-n}$.

Step 2. For some m, $t_m > C_2(r+\delta)^{-m}$ and $t_{m+3} > C_2(r+\delta)^{-(m+3)}$.

Step 3. For some N and all $n > N$ with $n-1$ a multiple of 3, $s_n > C_3(r+\delta)^{-n}$.

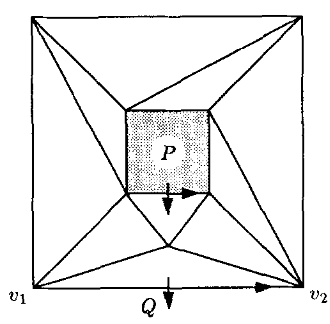

FIGURE 2

The theorem follows easily from Step 3, the earlier discussion, and the fact that for a power series

$$\limsup_{n \to \infty} s_n^{1/n} = 1/r.$$

Step 1 follows immediately from the lim sup result. We now give constructions to prove Steps 2 and 3.

Every map counted by t_n can be embedded in the face P of Fig. 1 to obtain a map counted by t_{n+e}. (It happens that $e = 63$.) By removing the diagonals of the 5-cycle A, placing a vertex x in its interior, and joining x to the vertices of the cycle, we obtain a map counted by t_{n+e+3}. Since these maps are injections, $t_{n+e} \geqslant t_n$ and $t_{n+e+3} \geqslant t_n$. By choosing C_1 sufficiently large, this proves Step 2.

By a simple counting argument, $m = 3k - 65$ for some integer k in Step 2. All integers exceeding $k(k+1)$ can be written as a linear combination of k and $k+1$ with non-negative integer coefficients. We now prove Step 3. Embed the map of Fig. 3 in that of Fig. 1 by identifying the cycles P to obtain a map M with root face Q. One can embed maps counted by s_{i-65} and s_{j-65} in the faces R and S of M, respectively, to obtain a map counted by s_{i+j-65}. By iterating this process, choosing C_2, and setting $N = 3k(k+1) + 65$, Step 3 follows.

4. Submaps

Our goal in this section is to prove the following theorem.

THEOREM 2. *Let $1 \leqslant k \leqslant 5$ and let T be any k-simple $\{3, k\}$-map where only the root face is not a triangle. There are numbers $c > 0$ and $d > 1$ such*

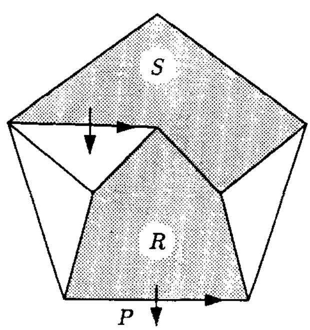

FIGURE 3

that for all sufficiently large n, the fraction of n edge k-simple triangulations that contain less than cn disjoint copies of T is at most d^{-n}.

By taking the dual, we obtain a result for cyclically k-connected cubic maps. The notion that corresponds to the $\{3, k\}$-map T being a submap is the dual of T being a submap that is attached to the rest of the map by removing its root vertex and attaching the k edges incident to it to the rest of the map.

The cases $k=2$ and $k=3$ were done by Richmond and Wormald [5]. The case $k=1$ can be done easily using the ideas in [1]. The cases $k=4$ and $k=5$ are somewhat more difficult. We prove them by showing that the Assumption in Theorem 2 of [1] is satisfied.

The basic idea is to select k-cycles in a k-simple triangulation, remove their diagonals, and insert copies of the map in Fig. 1 or Fig. 2, depending on whether $k=5$ or $k=4$. The map T is inserted in the faces P. The result can rather easily be shown to be a k-simple triangulation. (The 5-simple case is the least easy. A lemma of McCuaig [4] implies that we can insert the map in Fig. 1 since all of its boundary vertices have degree exceeding 3.) There are two difficulties with this procedure:

(i) Since *any* subset of our cycles could be chosen for this replacement, overlap must be avoided.

(ii) The procedure must be reversible.

We show how to take care of these problems for 5-simple maps. The process for 4-simple maps should then be obvious.

Let $\{v_1, v_2, v_3\}$ be the vertices of a triangle in a 5-simple triangulation and let x (resp., y) be the third vertex of the other triangle that has side v_2, v_3 (resp., $v_3 v_1$). The circuit $v_1 v_2 x v_3 y$ has no repeated vertices because multiple edges are forbidden. We want to select a subset of the patterns such that they have no triangles in common and do not contain the root edge of the map. A simple counting argument shows that any given triangle belongs to at most nine such patterns of three triangles. Thus, there is an integer k such that for any 5-simple triangulation T, we can choose our subset to be of size at least $[e(T)/k]$.

Reversibility requires that there be no overlap and that we be able to identify the way replacement was done. A simple check of Fig. 1 shows that no part of it could overlap with another part of Fig. 1: If, say Q_1 and Q_2 overlapped, the boundary of one, say Q_1, would contain a path of length 2 or 3 in the other and one of the internal vertices on this path would have degree 2 in Q_1, a contradiction. The asymmetry of the figure makes it possible to identify v_1 and v_2 in Fig. 1 and thus reverse the process of insertion.

5. Longest paths

Our goal in this section is to prove the following theorem.

Theorem 3. *Let $1 \leqslant k \leqslant 5$. There are constants $c<1$ and $d>1$ such that for all sufficiently large v, the fraction of v vertex cyclically k-connected cubic maps that have a path with length exceeding cv is at most d^{-v}.*

The cases $k=2, 3$ of the theorem were proved in [5]. We will prove $k=4$ and $k=5$. The case $k=1$ is much easier. Note that the number of edges is $3v/2$, so we can work with number of edges if we choose.

The idea is to apply the dual of Theorem 2. The "nearly cubic" map M that we insert will be such that any path that neither begins nor ends in the copy of M will miss at least one vertex in that copy of M. If there are at least $c'v$ copies of M in a map, then the longest path length is at most $(v-1)-(c'v-2)$.

To complete the proof, we need to exhibit M. For $k=4$, we could use a construction due to Tutte [7], but it is simpler to adapt Walther's [11] construction. If one removes the five outside edges from his Fig. 3, one obtains a map fragment such that a path in the map that contains it omits at least one of its vertices. Thus, the longest path in any map containing j disjoint copies of the fragment must omit at least j vertices. The fragment can be used directly for $k=5$. One can adapt it to $k=4$ by replacing the upper path of length 2 that joins G_1^2 and G_1^3 by a single edge.

References

1. E. A. Bender, Z.-C. Gao, and L. B. Richmond, Submaps of maps. I. General 0–1 laws, *J. Combin. Theory Ser. B* **55** (1992), 104–117.
2. Z.-C. Gao. The number of rooted triangular maps on a surface, *J. Combin. Theory Ser. B* **52** (1991), 236–249.
3. B. Grünbaum, "Convex Polytopes," Wiley, London, 1967.
4. W. D. Mc Cuaig, "Edge Reductions in Cyclic k-Connected Cubic Graphs," Ph.D. thesis, University of Waterloo, 1987.
5. L. B. Richmond and N. C. Wormald, On random triangulations of the plane, *European J. Combin.* **9** (1988), 61–71.
6. P. G. Tait, Listings topologie, *Philos. Mag. (5)* **17** (1884), 30–36; *Sci. Papers* **2**, 85–98.
7. W. T. Tutte, On Hamiltonian circuits, *J. London. Math. Soc.* **21** (1946), 98–101.
8. W. T. Tutte, A non-Hamiltonian planar graph, *Acta Math. Acad. Sci. Hungar.* **11** (1960), 371–375.
9. W. T. Tutte, A census of planar triangulations, *Canad. J. Math.* **14** (1962), 21–38.
10. H. Walther, Ein kubischer, planarer, zyklisch fünffach zusammenhängender Graph, der keinen Hamiltonkreis besistzt, *Wiss. Z. Tech. Hochsch. Ilmenau* **11** (1965), 163–166.
11. H. Walther, On the problem of the existence of Hamilton-lines in planar regular graphs, *in* "Colloquium on Graph Theory, Tihany" (P. Erdős and G. Katona, Eds.), pp. 341–343, Academic Press, San Diego, CA, 1968.